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PSL Course Packet

MATH 110

Precalculus Review

Produced in collaboration with the Penn State Department of Mathematics

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PennState
Undergraduate Education

Algebraic Simplification Techniques

Simplifying Expressions Involving Fractions

- **Adding or Subtracting Fractions**

  If the fractions have a common denominator, then they can be combined into a single fraction by adding or subtracting the numerators accordingly.

  $$\boxed{\frac{A}{B} \pm \frac{C}{B} = \frac{A \pm C}{B}} \qquad \text{e.g., } \frac{2}{3} + \frac{5}{3} = \frac{7}{3}$$

  If the fractions do not have a common denominator, then a common denominator can be formed by multiplying the denominators together.

  $$\boxed{\frac{A}{B} \pm \frac{C}{D} = \frac{AD}{BD} \pm \frac{BC}{BD} = \frac{AD \pm BC}{BD}} \qquad \text{e.g., } \frac{2}{3} - \frac{5}{7} = \frac{2 \cdot 7 - 3 \cdot 5}{3 \cdot 7} = -\frac{1}{21}$$

- **Multiplying Fractions**

  To multiply two fractions, multiply the numerators and multiply the denominators of the two fractions.

  $$\boxed{\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}} \qquad \text{e.g., } \frac{2}{3} \times \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$$

- **Dividing Fractions**

  To divide two fractions, remember that division is the same as multiplication by the reciprocal. In other words, dividing by C/D is the same as multiplying by D/C.

  $$\boxed{\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C} = \frac{AD}{BC}} \qquad \text{e.g., } \frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15}$$

Example 1. Rewrite $\dfrac{x+1}{x+4} - \dfrac{3x-2}{x+4}$ as a single ratio.

Step 1. Since the ratios already have the same denominator, we need only apply the formula for subtracting fractions with a common denominator.

$$\frac{A}{B} - \frac{C}{B} = \frac{A-C}{B}$$

Step 2. Apply the formula from **Step 1** with $A = x+1$, $B = x+4$, and $C = 3x-2$.

$$\begin{aligned}
\frac{x+1}{x+4} - \frac{3x-2}{x+4} &= \frac{x+1-(3x-2)}{x+4} && \text{Subtract numerators} \\
&= \frac{x+1-3x+2}{x+4} && \text{Distribute the minus sign} \\
&= \frac{3-2x}{x+4} && \text{Combine like terms}
\end{aligned}$$

Example 2. Rewrite $\dfrac{3}{x} + \dfrac{4}{x-5}$ as a single ratio.

Step 1. Since the ratios do not have the same denominator, we will apply the formula for adding fractions with different denominators.

$$\frac{A}{B} + \frac{C}{D} = \frac{AD}{BD} + \frac{BC}{BD} = \frac{AD+BC}{BD}$$

Step 2. Apply the formula from **Step 1** with $A = 3$, $B = x$, $C = 4$, and $D = x-5$.

$$\begin{aligned}
\frac{3}{x} + \frac{4}{x-5} &= \frac{3(x-5)}{x(x-5)} + \frac{4x}{x(x-5)} && \text{Get a common denominator} \\
&= \frac{3(x-5)+4x}{x(x-5)} && \text{Add numerators} \\
&= \frac{3x-15+4x}{x(x-5)} && \text{Distribute the 3} \\
&= \frac{7x-15}{x(x-5)} && \text{Combine like terms}
\end{aligned}$$

Example 3. Simplify

$$\frac{\left(\dfrac{3p}{\sqrt{180-6p}}\right)}{\sqrt{180-6p}}.$$

Step 1. Recall the formula for the quotient of fractions.

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C} = \frac{AD}{BC}$$

Step 2. Use the formula from **Step 1** to rewrite the equation.

$$\begin{aligned}
\frac{\left(\dfrac{3p}{\sqrt{180-6p}}\right)}{\sqrt{180-6p}} &= \frac{3p}{\sqrt{180-6p}} \div \sqrt{180-6p} \\
&= \frac{3p}{\sqrt{180-6p}} \times \frac{1}{\sqrt{180-6p}} && \text{Division is multiplication by reciprocal} \\
&= \frac{3p}{\sqrt{180-6p} \times \sqrt{180-6p}} && \text{Multiply numerators and denominators} \\
&= \frac{3p}{180-6p} && \text{Simplify}
\end{aligned}$$

Distributive Property of Multiplication

The distributive property of multiplication can be used to rewrite a product (where at least one factor is a sum) as a sum.

$$\boxed{A(B+C) = AB + AC} \qquad \text{e.g., } 7(3+2) = 7\cdot 3 + 7\cdot 2$$

Example 4. Expand $x^2(5x^3+7)$.

Step 1. Use the distributive property of multiplication to expand the given product.

$$\begin{aligned} x^2(5x^3+7) &= 5x^2x^3 + 7x^2 && \text{Distribute } x^2 \\ &= 5x^5 + 7x^2 && \text{Simplify} \end{aligned}$$

The FOIL Method

The FOIL method is a way to remember how to apply the distributive property of multiplication when expanding the product of two binomial expressions. FOIL is an acronym for

- **F**irst (i.e., multiply the first terms from each binomial)
- **O**uter (i.e., multiply the first term from the first factor and the second term from the second factor)
- **I**nner (i.e., multiply the second term from the first factor and the first term from the second factor)
- **L**ast (i.e., multiply the second terms of each binomial)

$$\boxed{(a+b)(c+d) = \underbrace{ac}_{\textbf{F}\text{irst}} + \underbrace{ad}_{\textbf{O}\text{uter}} + \underbrace{bc}_{\textbf{I}\text{nner}} + \underbrace{bd}_{\textbf{L}\text{ast}}}$$

Example 5. Expand $(x+2)(3x-5)$ using the FOIL method.

Step 1. Recall the formula for the FOIL method.

$$(a+b)(c+d) = ac + ad + bc + bd$$

Step 2. Use the FOIL method to expand the expression.

$$\begin{aligned} (x+2)(3x-5) &= x(3x) + x(-5) + 2(3x) + 2(-5) && \text{FOIL} \\ &= 3x^2 - 5x + 6x - 10 && \text{Simplify each term} \\ &= 3x^2 + x - 10 && \text{Combine like terms} \end{aligned}$$

Squaring a Binomial

When applying the FOIL method to the square of a binomial (i.e., $(a+b)^2$ or $(a-b)^2$), we arrive at the following formulas:

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2} \qquad \boxed{(a-b)^2 = a^2 - 2ab + b^2}$$

Example 6. Expand $(3x-5)^2$ by squaring the binomial.

Step 1. Apply $(a-b)^2 = a^2 - 2ab + b^2$ with $a = 3x$ and $b = 5$.

$$\begin{aligned} (3x-5)^2 &= (3x)^2 - 2(3x)(5) + 5^2 && \text{Square the binomial} \\ &= 3^2x^2 - 30x + 25 && \text{Simplify} \\ &= 9x^2 - 30x + 25 \end{aligned}$$

Example 7. Expand $3x^5(4+x)^2$.

Step 1. Apply $(a+b)^2 = a^2 + 2ab + b^2$ with $a = 4$ and $b = x$.

$$\begin{aligned} 3x^5(4+x)^2 &= 3x^5(4^2 + 2(4)(x) + x^2) && \text{Square the binomial} \\ &= 3x^5(16 + 8x + x^2) && \text{Simplify} \end{aligned}$$

Step 2. Use the distributive property of multiplication.

$$\begin{aligned} 3x^5(16 + 8x + x^2) &= 3x^5(16) + 3x^5(8x) + 3x^5(x^2) && \text{Distribute } 3x^5 \\ &= 48x^5 + 24x^6 + 3x^7 && \text{Simplify} \end{aligned}$$

Factoring Techniques

Pull out Common Factors

The act of pulling out common factors from a sum can be thought of as applying the distributive property of multiplication in reverse.

$$\boxed{AB + AC = A(B + C)} \qquad \text{e.g., } 21{+}14 = 7{\cdot}3{+}7{\cdot}2 = 7(3{+}2)$$

Example 8. Factor

$$10x^5 + 15x^4$$

by pulling out common factors.

Step 1. Determine the factors common to both terms.

- Constant factors: 10 and 15 are both multiples of 5.
- Powers of x: the smallest power of x is 4.

Therefore, the factor common to both terms is $5x^4$.

Step 2. Pull out the common factor of $5x^4$.

$$\begin{aligned} 10x^5 + 15x^4 &= 5x^4 \cdot 2x + 5x^4 \cdot 3 && \text{Write each term as a multiple of } 5x^4 \\ &= 5x^4(2x + 3) && \text{Pull out common factor} \end{aligned}$$

Example 9. Factor

$$4x^3(2x^3 + 1)^5 + 30x^6(2x^3 + 1)^4$$

by first pulling out common factors.

Step 1. Determine the factors common to both terms.

- Constant factors: 4 and 30 are both multiples of 2.
- Powers of x: the smallest power of x is 3.
- Powers of $2x^3 + 1$: the smallest power of $2x^3 + 1$ is 4.

Therefore, the factor common to both terms is $2x^3(2x^3 + 1)^4$.

Step 2. Pull out the common factor of $2x^3(2x^3 + 1)^4$.

$$\begin{aligned} &4x^3(2x^3 + 1)^5 + 30x^6(2x^3 + 1)^4 \\ &= 2x^3(2x^3 + 1)^4 \cdot 2(2x^3 + 1) + 2x^3(2x^3 + 1)^4 \cdot 15x^3 \\ &= 2x^3(2x^3 + 1)^4[2(2x^3 + 1) + 15x^3] && \text{Pull out common factor} \\ &= 2x^3(2x^3 + 1)^4(4x^3 + 2 + 15x^3) && \text{Simplify expression inside [ ]} \\ &= 2x^3(2x^3 + 1)^4(19x^3 + 2) \end{aligned}$$

Example 10. Factor

$$4x^3(2x^3+1)^{1/2}+3x^6(2x^3+1)^{-1/2}$$

by first pulling out common factors.

Step 1. Determine the factors common to both terms.

- Constant factors: 4 and 3 do not have any factors in common.
- Powers of x: the smallest power of x is 3.
- Powers of $2x^3+1$: the smallest power of $2x^3+1$ is $-1/2$.

Therefore, the factor common to both terms is $x^3(2x^3+1)^{-1/2}$.

Step 2. Pull out the common factor of $x^3(2x^3+1)^{-1/2}$.

$4x^3(2x^3+1)^{1/2}+3x^6(2x^3+1)^{-1/2}$

$= x^3(2x^3+1)^{-1/2}\cdot 4(2x^3+1)+x^3(2x^3+1)^{-1/2}\cdot 3x^3$

$= x^3(2x^3+1)^{-1/2}[4(2x^3+1)+3x^3]$ Pull out common factor

$= x^3(2x^3+1)^{-1/2}(8x^3+4+3x^3)$ Simplify expression inside [ ]

$= x^3(2x^3+1)^{-1/2}(11x^3+4)$ Combine like terms

$= \dfrac{x^3(11x^3+4)}{\sqrt{2x^3+1}}$

Difference of Squares

When applying the FOIL technique to a product of the form $(A+B)(A-B)$, we get the following result.

$$\begin{aligned}(A+B)(A-B) &= A\cdot A - A\cdot B + B\cdot A - B\cdot B\\ &= A^2 \cancel{-AB} \cancel{+AB} - B^2\\ &= A^2 - B^2\end{aligned}$$

We refer to $A^2 - B^2$ as a difference of squares and can reinterpret the above calculation as a way to factor any difference of squares.

$$\boxed{A^2 - B^2 = (A+B)(A-B)}$$

Example 11. Factor $x^2 - 25$.

Step 1. Rewrite the expression as a difference of squares.

$$x^2 - 25 = x^2 - 5^2$$

Step 2. Apply the difference of squares formula with $A = x$ and $B = 5$.

$$\begin{aligned}x^2 - 25 &= x^2 - 5^2 && \textbf{Step 1}\\ &= (x+5)(x-5) && \text{Difference of squares}\end{aligned}$$

Example 12. Factor $9x^3 - 4x^5$.

Step 1. Determine the factors common to both terms.

- Constant factors: 9 and -4 do not have any factors in common.
- Powers of x: the smallest power of x is 3.

Therefore, the factor common to both terms is x^3.

Step 2. Pull out the common factor of x^3.

$$\begin{aligned}9x^3 - 4x^5 &= x^3(9) - x^3(4x^2)\\ &= x^3(9 - 4x^2) && \text{Pull out common factor}\end{aligned}$$

Step 3. Rewrite $9 - 4x^2$ as a difference of squares.

$$x^3(9 - 4x^2) = x^3[3^2 - (2x)^2]$$

Step 4. Apply the difference of squares formula with $A = 3$ and $B = 2x$.

$$\begin{aligned}9x^3 - 4x^5 &= x^3(9 - 4x^2) && \textbf{Step 2}\\ &= x^3[3^2 - (2x)^2] && \textbf{Step 3}\\ &= x^3(3+2x)(3-2x) && \text{Difference of squares}\end{aligned}$$

AC Grouping Method

The AC grouping method is a technique for factoring certain quadratic expressions of the form

$$ax^2 + bx + c.$$

The process is broken down into the following five steps.

1. Find two integers, r and s, that multiply to $a \times c$ and sum to b.
2. Replace bx with $rx + sx$.
3. Group terms with common factors.
4. Pull out common factors from each group.
5. Pull out common factor.

Example 13. Factor $6x^2 + 7x - 5$ using the AC grouping method.

Step 1. Find two integers that multiply to $6(-5) = -30$ and sum to 7.

Since the product is negative, the two numbers must have opposite signs. And since the sum is positive, the larger number in absolute value, must be positive.

Product equals -30	Sum equals 7?	
$-1 \times 30 = -30$	$-1 + 30 = 29$	NO
$-2 \times 15 = -30$	$-2 + 15 = 13$	NO
$-3 \times 10 = -30$	$-3 + 10 = 7$	YES

Step 2. Since $7 = -3 + 10$, replace the linear term, $7x$, with $-3x + 10x$.

$$6x^2 - 3x + 10x - 5$$

Step 3. Group terms with common factors.

$$(6x^2 - 3x) + (10x - 5)$$

Step 4. Pull out common factors from each group.

$$3x(2x - 1) + 5(2x - 1)$$

Step 5. Pull out common factor of $2x - 1$.

$$(2x - 1)(3x + 5)$$

Therefore,

$$6x^2 + 7x - 5 = (2x - 1)(3x + 5)$$

Check Your Work: After factoring a polynomial, it's always a good idea to check your work by expanding the product using the distributive property of multiplication and/or the FOIL method:

$$\begin{aligned}
(2x - 1)(3x + 5) &= (2x)(3x) + (2x)(5) + (-1)(3x) + (-1)(5) && \text{FOIL} \\
&= 6x^2 + 10x - 3x - 5 && \text{Simplify} \\
&= 6x^2 + 7x - 5 && \text{Combine like terms}
\end{aligned}$$

Example 14. Factor $x^2 - 13x + 36$ using the AC grouping method

Step 1. Find two integers that multiply to $1 \times 36 = 36$ and sum to -13.

Since the product is positive, the two numbers must have the same sign. And since the sum is negative, both numbers must be negative.

Product equals 36	Sum equals -13?
-1×-36	NO
-2×-18	NO
-3×-12	NO
-4×-9	YES

Step 2. Since $-13 = -4 - 9$, replace the linear term, $-13x$, with $-4x - 9x$.

$$x^2 - 4x - 9x + 36$$

Step 3. Group terms with common factors.

$$(x^2 - 4x) + (-9x + 36)$$

Step 4. Pull out common factors from each group.

$$x(x - 4) + (-9)(x - 4)$$

Step 5. Pull out common factor of $x - 4$.

$$(x - 4)(x - 9)$$

Therefore,

$$x^2 - 13x + 36 = (x - 4)(x - 9)$$

Check Your Work:

$$\begin{aligned} (x-4)(x-9) &= x^2 - 9x - 4x + 36 && \text{FOIL} \\ &= x^2 - 13x + 36 && \text{Combine like terms} \end{aligned}$$

A Special Case of the AC Grouping Method

If the coefficient of x^2 is one (i.e., $a = 1$ in $ax^2 + bx + c$), then Step 1 of the *AC* grouping method is to find two numbers that multiply to c and sum to b. Once these numbers have been found, then the factorization can be written as

$$x^2 + bx + c = (x + r)(x + s)$$

where $r + s = b$ and $rs = c$.

Example 15. Factor $x^2 - 4x - 12$.

Step 1. Find two integers that multiply to -12 and sum to -4.

Since the product is negative, the two numbers must have opposite signs. And since the sum is negative, the larger number in absolute value, must be negative.

Product equals -12	Sum equals -4?
1×-12	NO
2×-6	YES

Step 2. Since the coefficient of x^2 is one, then the factorization is given by

$$x^2 - 4x - 12 = (x+2)(x-6).$$

Check Your Work:

$$\begin{aligned} (x+2)(x-6) &= x^2 - 6x + 2x - 12 && \text{FOIL} \\ &= x^2 - 4x - 12 && \text{Combine like terms} \end{aligned}$$

Example 16. Factor $7x^4 + 35x^3 + 42x^2$.

Step 1. Determine the factors common to both terms.

- Constant factors: 7, 35, and 42 are all multiples of 7.
- Powers of x: the smallest power of x is 2.

Therefore, the factor common to all three terms is $7x^2$.

Step 2. Pull out the common factor of $7x^2$.

$$\begin{aligned} 7x^4 + 35x^3 + 42x^2 &= 7x^2(x^2) + 7x^2(5x) + 7x^2(6) \\ &= 7x^2(x^2 + 5x + 6) && \text{Pull out common factor} \end{aligned}$$

Step 3. Factor $x^2 + 5x + 6$ by finding two integers that multiply to 6 and sum to 5.

Product equals 6	Sum equals 5?
1×6	NO
2×3	YES

Therefore,

$$\begin{aligned} 7x^4 + 35x^3 + 42x^2 &= 7x^2(x^2 + 5x + 6) && \textbf{Step 2} \\ &= 7x^2(x+2)(x+3) && \text{AC grouping} \end{aligned}$$

Check Your Work:

$$\begin{aligned} 7x^2(x+2)(x+3) &= 7x^2(x^2 + 3x + 2x + 6) && \text{FOIL} \\ &= 7x^2(x^2 + 5x + 6) && \text{Combine like terms} \\ &= 7x^2(x^2) + 7x^2(5x) + 7x^2(6) && \text{Distribute } 7x^2 \\ &= 7x^4 + 35x^3 + 42x^2 && \text{Simplify} \end{aligned}$$

Solving Equations

How to Solve an Equation

To find all values of x that satisfy an equation (e.g., $f(x) = g(x)$), complete the following steps:

- Rewrite equation as $f(x) - g(x) = 0$, if needed.
- Simplify and factor the left-hand side.
- Set each factor equal to zero and solve for x.

Example 17. Find all values of x such that $x^2 - 4x - 12 = 0$.

Step 1. Factor $x^2 - 4x - 12$.

Recall from **Example 15** that

$$x^2 - 4x - 12 = (x + 2)(x - 6)$$

Step 2. Set each factor equal to zero and solve for x.

$$\begin{aligned} x + 2 = 0 \quad &\Rightarrow \quad x = -2 \\ x - 6 = 0 \quad &\Rightarrow \quad x = 6 \end{aligned}$$

Therefore, $x = 6$ and $x = -2$ satisfy $x^2 - 4x - 12 = 0$.

Check Your Work:

$$\begin{aligned} (-2)^2 - 4(-2) - 12 &= 4 + 8 - 12 = 0 \\ 6^2 - 4(6) - 12 &= 36 - 24 - 12 = 0 \end{aligned}$$

Example 18. Find all values of p such that $\dfrac{3p}{180 - 6p} = 1$.

Step 1. Multiply both sides of $\frac{3p}{180-6p} = 1$ by the denominator, $180 - 6p$.

$$3p = 180 - 6p$$

Step 2. Subtract $180 - 6p$ from both sides.

$$\begin{aligned} 3p - (180 - 6p) &= 0 \\ 3p - 180 + 6p &= 0 && \text{Distribute the minus sign} \\ 9p - 180 &= 0 && \text{Combine like terms} \end{aligned}$$

Step 3. Solve for p.

$$\begin{aligned} 9p &= 180 && \text{Add 180 to both sides} \\ p &= 20 && \text{Divide both sides by 9} \end{aligned}$$

Therefore, $p = 20$ is the only value that satisfies $\frac{3p}{180-6p} = 1$.

Check Your Work:

$$\frac{3(20)}{180-6(20)} = \frac{60}{180-120} = \frac{60}{60} = 1$$

Example 19. Find all points of intersection of $f(x) = 6x^2 - 4x$ and $g(x) = 2 - 5x$.

Step 1. Set $f(x) = g(x)$.

Points of intersection can be found by setting the two curves equal to each other and solving for x.

$$6x^2 - 4x = 2 - 5x$$

Step 2. Subtract $2 - 5x$ from both sides of the equation in **Step 1**.

$6x^2 - 4x - (2 - 5x) = 0$	Note the parentheses around $2 - 5x$
$6x^2 - 4x - 2 + 5x = 0$	Distribute the minus sign
$6x^2 + x - 2 = 0$	Combine like terms

Step 3. Use the AC grouping method to factor $6x^2 + x - 2$.

Find two integers that multiply to $6(-2) = -12$ and sum to 1.

Product equals -12	Sum equals 1?
-1×12	NO
-2×6	NO
-3×4	YES

Therefore,

$$\begin{aligned} 6x^2 + x - 2 &= 6x^2 - 3x + 4x - 2 \\ &= 3x(2x-1) + 2(2x-1) \\ &= (3x+2)(2x-1) \end{aligned}$$

Step 4. Set each factor equal to zero and solve for x.

$$\begin{array}{lclcl} 3x + 2 = 0 & \Rightarrow & 3x = -2 & \Rightarrow & x = -2/3 \\ 2x - 1 = 0 & \Rightarrow & 2x = 1 & \Rightarrow & x = 1/2 \end{array}$$

Therefore, the only points of intersection of $f(x) = 6x^2 - 4x$ and $g(x) = 2 - 5x$ occur when $x = -2/3$ and $x = 1/2$.

Check Your Work:

x	$f(x) = 6x^2 - 4x$	$g(x) = 2 - 5x$
$-2/3$	$6(-2/3)^2 - 4(-2/3) = 16/3$	$2 - 5(-2/3) = 16/3$
$1/2$	$6(1/2)^2 - 4(1/2) = -1/2$	$2 - 5(1/2) = -1/2$

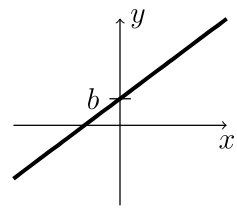
Graphing

Equations for a Line

- Slope-Intercept Form

  The equation of the line with slope m and y-intercept equal to b is

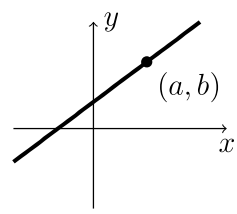
  $$\boxed{y = mx + b}$$



- Point-Slope Form

  The equation of the line with slope m that goes through the point (a, b) is

  $$\boxed{y - b = m(x - a)}$$

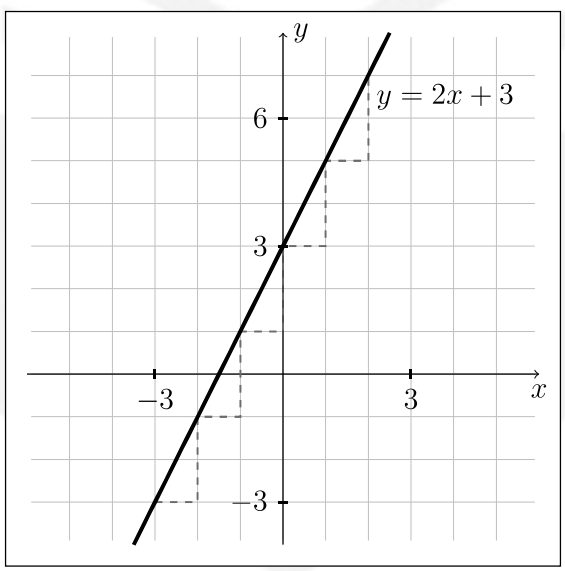


Recall that a positive slope means that the line goes up from left-to-right and a negative slope means that the line goes down from left-to-right.

Example 20. Sketch the graph of the line defined by $y = 2x + 3$.

Step 1. Since $y = 2x + 3$ is in slope-intercept form, the line has slope 2 and a y-intercept of 3.

Step 2. Draw the line with slope equal to 2 and y-intercept equal to 3.

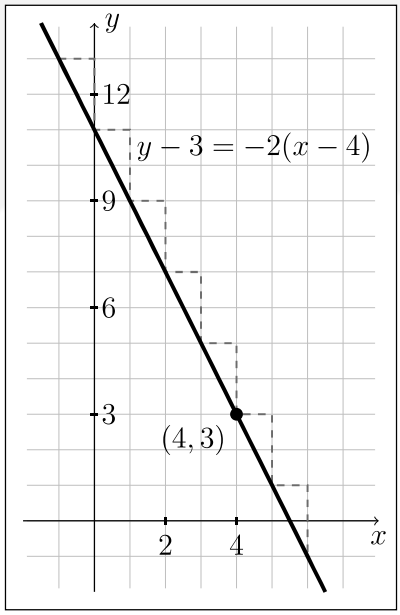


Note that the red dashed lined is not part of the graph and is used only as a guide for drawing a line with slope 2. In particular, in order for a line to have slope equal to 2, if the x-coordinate of any point on the line is increased by 1 unit, then the y-coordinate must be increased by 2 units.

Example 21. Sketch the graph of the line defined by $y - 3 = -2(x - 4)$.

Step 1. Since $y - 3 = -2(x - 4)$ is in point-slope form, the line has slope -2 and goes through the point $(4, 3)$.

Step 2. Draw the line with slope equal to -2 that goes through the point $(4, 3)$.

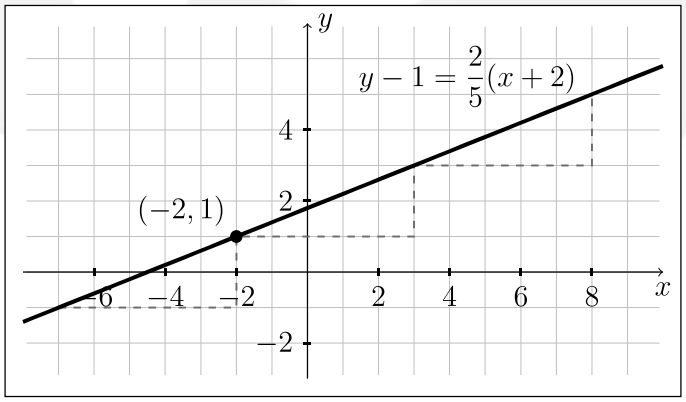


In order for a line to have slope equal to -2, if the x-coordinate of any point on the line is increased by 1 unit, then the y-coordinate must be decreased by 2 units.

Example 22. Sketch the graph of the line defined by $y - 1 = \dfrac{2}{5}(x + 2)$.

Step 1. Since $y - 1 = \frac{2}{5}(x + 2)$ is in point-slope form, the line has slope $2/5$ and goes through the point $(-2, 1)$.

Step 2. Draw the line with slope equal to $2/5$ that goes through the point $(-2, 1)$.



In order for a line to have slope equal to $2/5$, if the x-coordinate of any point on the line is increased by 5 units, then the y-coordinate must be increased by 2 units.

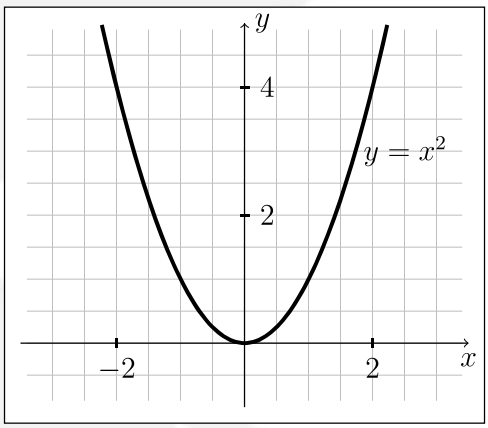
Graphing Quadratic Polynomials

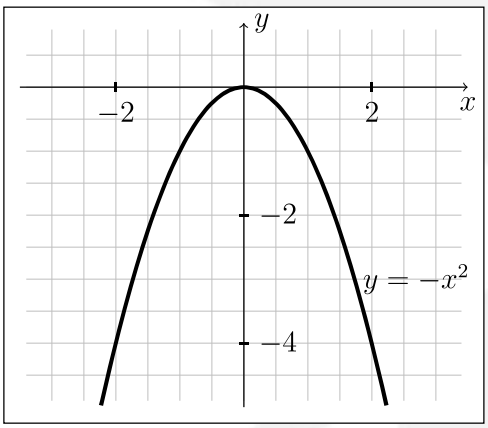
The general form of a quadratic polynomial (i.e., a polynomial of degree two) is

$$y = ax^2 + bx + c$$

where a, b, and c are real numbers and $a \neq 0$. The graph of a quadratic polynomial has the shape of a parabola. If $a > 0$, then the parabola opens upward (i.e., looks like the letter "U") and if $a < 0$, then the parabola opens downward.

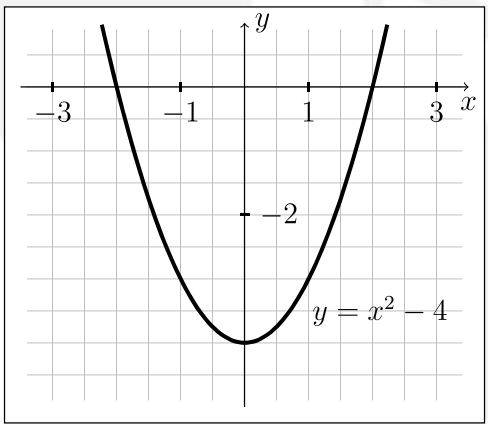
Example 23. Compare the graphs of $y = x^2$ and $y = -x^2$.

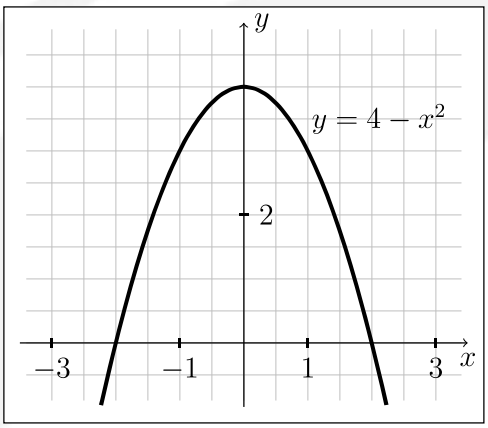




Notice how the graph of $y = x^2$ is a parabola that goes through the point $(0, 0)$ and opens upward while the graph of $y = -x^2$ is a parabola that also goes through the point $(0, 0)$ but opens downward.

Example 24. Compare the graphs of $y = x^2 - 4$ and $y = 4 - x^2$





Notice how the graph of $y = x^2 - 4$ looks like the graph of $y = x^2$ with each point shifted down 4 units. Also, the graph of $y = 4 - x^2$ looks like the graph of $y = -x^2$ with each point shifted up 4 units.

Example 25. Sketch the graph of $f(x) = x^2 - 4x - 12$.

Step 1. Determine the y-intercept by evaluating $f(0)$.

$$\begin{aligned} f(0) &= 0^2 - 4(0) - 12 \\ &= -12 \end{aligned}$$

Therefore the graph of $y = f(x)$ goes through the point $(0, -12)$.

Step 2. Determine the x-intercepts by setting $f(x) = 0$ and solving for x.

Recall from **Examples 15 and 17** that

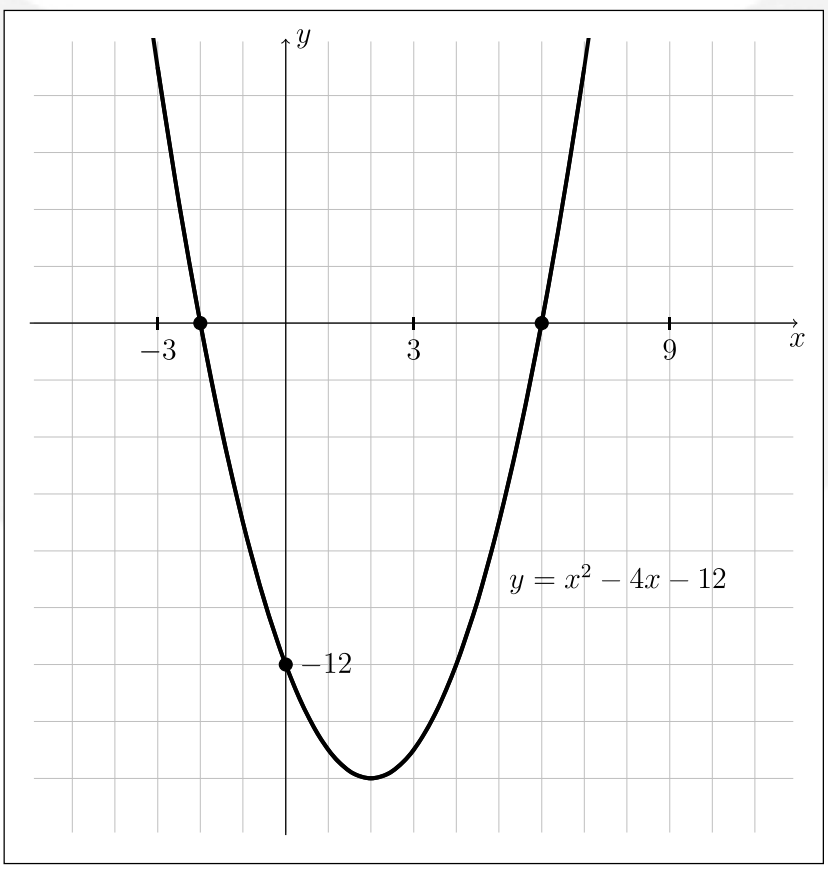
$$x^2 - 4x - 12 = (x + 2)(x - 6)$$

Now set each factor equal to zero and solve for x.

$$\begin{aligned} x + 2 = 0 \quad &\Rightarrow \quad x = -2 \\ x - 6 = 0 \quad &\Rightarrow \quad x = 6 \end{aligned}$$

Therefore the graph of $y = f(x)$ goes through the points $(-2, 0)$ and $(6, 0)$.

Step 3. Draw the graph of a parabola that opens upward (since the coefficient of x^2 in $f(x)$ is positive) and goes through the points found in **Steps 1** and **2**.



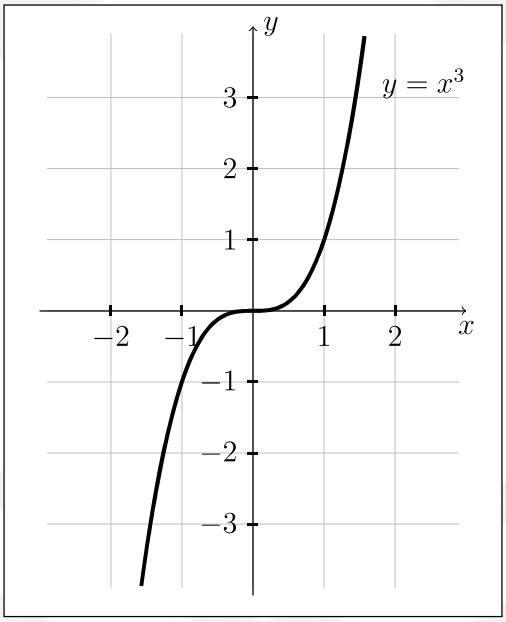
Graphing Power and Root Functions

Any function of the form

$$y = x^r$$

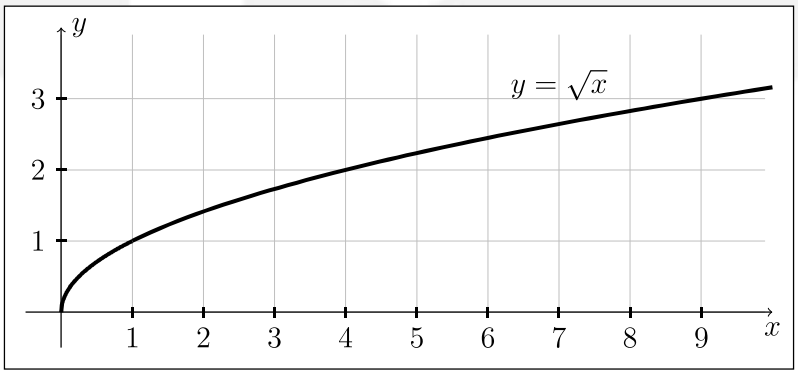
where r is any real number is called a power function. Thus x^2, x^3, x^4, etc. are examples of power functions. Root functions, like the square root (i.e., $\sqrt{x}$ or $x^{1/2}$) and cube root (i.e., $\sqrt[3]{x}$ or $x^{1/3}$) are also examples of power functions.

Example 26. Sketch the graph of $y = x^3$.



Notice how the graph of $y = x^3$ always increases from left-to-right and looks like a horizontal line as it goes through the origin.

Example 27. Sketch the graph of the square root function, $y = \sqrt{x}$.



Notice how the graph of $y = \sqrt{x}$ looks like the upper half of a parabola that opens to the right.

Solving Inequalities

How to Solve an Inequality

In order to find all values of x such that $f(x) > 0$ or $f(x) < 0$, use the following procedure.

- Find all values of x such that $f(x) = 0$ or $f(x)$ is not defined.
- Use the values found to break up the number line into intervals and select one number from each interval to plug into $f(x)$ to determine if $f(x)$ is positive or negative on that interval.

  OR

  Use the values found to help draw the graph of $f(x)$. Portions of the graph that are above the x-axis correspond to values of x where $f(x) > 0$ while portions of the graph that are below the x-axis correspond to values of x where $f(x) < 0$.

Example 28. Find all values of x such that $x^2 + 2x - 3 > 0$.

Step 1. Find all values of x such that $x^2 + 2x - 3 = 0$.

Use the AC method to factor $x^2 + 2x - 3$.

$$x^2 + 2x - 3 = (x + 3)(x - 1)$$

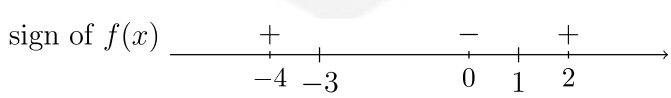
since 3 and -1 are two numbers that multiply to -3 and sum to 2. Now set each factor equal to zero and solve for x.

$$\begin{aligned} x + 3 = 0 \quad &\Rightarrow \quad x = -3 \\ x - 1 = 0 \quad &\Rightarrow \quad x = 1 \end{aligned}$$

Step 2. Use one of the following two methods to solve the inequality.

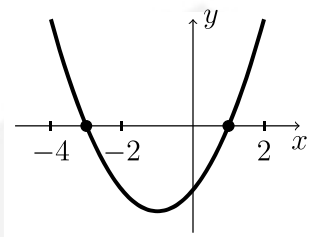
Method 1: Use the values of x found in **Step 1** to break up the number line and plug in one value from each interval into $f(x) = (x + 3)(x - 1)$.

Since $f(-3) = 0$ and $f(1) = 0$, pick one value less -3, one value between -3 and 1, and one value greater than 1. For example, since $f(-4) = 5 > 0$, $f(x) > 0$ for all $x < -3$. And since $f(2) = 5 > 0$, $f(x) > 0$ for all $x > 1$. However, $f(0) = -3 < 0$, and therefore $f(x) < 0$ for all $-3 < x < 1$. These calculations are summarized in the following diagram.



Therefore, $x^2 + 2x - 3 > 0$ whenever $x < -3$ or $x > 1$.

Method 2: Sketch the graph of $y = x^2 + 2x - 3$ by drawing a parabola that opens upward and goes through the points $(-3, 0)$ and $(1, 0)$.



Finding values of x such that $x^2+2x-3 > 0$ is equivalent to identifying the portions of the graph of $x^2 + 2x - 3$ that are above the x-axis. Based on the graph shown above, $x^2 + 2x - 3 > 0$ whenever $x < -3$ or $x > 1$.

Example 29. Find all values of x such that $\dfrac{x^2(x^2+3)}{(4-x^2)^3} < 0$.

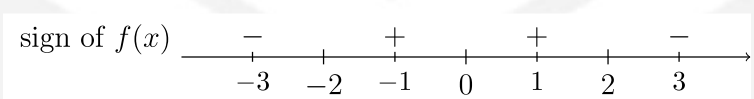
Step 1. Find all values of x such that $\frac{x^2(x^2+3)}{(4-x^2)^3} = 0$ or $\frac{x^2(x^2+3)}{(4-x^2)^3}$ is not defined.

$\frac{x^2(x^2+3)}{(4-x^2)^3} = 0$ whenever the numerator is equal to zero, which only happens when $x = 0$ (since $x^2 + 3$ is never equal to zero).

$\frac{x^2(x^2+3)}{(4-x^2)^3}$ is not defined whenever the denominator is equal to zero, which only happens when $x = -2$ or $x = 2$.

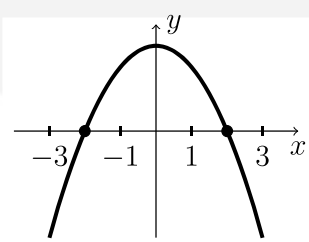
Step 2. Use one of the following two methods to solve the inequality.

Method 1: Break up the number line at $x = -2$, $x = 0$, and $x = 2$ and plug in one value from each interval to determine the sign of $f(x) = \frac{x^2(x^2+3)}{(4-x^2)^3}$ on that interval.



Therefore, $\frac{x^2(x^2+3)}{(4-x^2)^3} < 0$ when $x < -2$ or $x > 2$.

Method 2: Notice that x^2 and $x^2 + 3$ are both positive for $x \neq 0$, therefore $\frac{x^2(x^2+3)}{(4-x^2)^3} < 0$ whenever $4 - x^2 < 0$.



The graph of $y = 4 - x^2$ as shown above is below the x-axis (i.e., $4 - x^2 < 0$) if $x < -2$ or $x > 2$. Therefore, $\frac{x^2(x^2+3)}{(4-x^2)^3} < 0$ when $x < -2$ or $x > 2$.

The Domain of a Function

Definition 1. *The domain of a function is the set of all values of x for which the function is defined.*

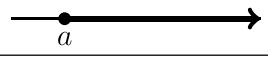
When determining the domain of a given function, do not include any value that leads to one or more of the following:

- division by zero
- the square root of a negative number
- the logarithm of zero or a negative number

Students should be familiar with the concepts of division and square roots. Logarithms will be discussed in class and in a subsequent PSL Course Packet. While there are other functions that lead to restrictions on the domain, we will limit our discussion to division, square roots, and logarithms.

Interval Notation

The domain of a function is typically written as a union of intervals. In this course, we will make use of interval notation to express domains. This notation is summarized in the following table.

Inequality	Number Line	Interval Notation
$a \leq x \leq b$		$[a, b]$
$a \leq x < b$		$[a, b)$
$a < x \leq b$		$(a, b]$
$a < x < b$		(a, b)
$a \leq x$		$[a, \infty)$
$a < x$		(a, ∞)
$x \leq b$		$(-\infty, b]$
$x < b$		$(-\infty, b)$

Given two intervals, A and B, then the union of A and B, denoted $A \cup B$, represents the collection of values that are in A or in B. For example,

$$(-\infty, 5) \cup [7, 10)$$

represents the values that are less than 5 (i.e., $x < 5$) or greater than or equal to 7 and less than 10 (i.e., $7 \leq x < 10$).

Example 30. Use interval notation to describe the domain of $1/x$.

Step 1. Describe the domain of $1/x$ using an inequality.

The domain of $1/x$ includes all real numbers except $x = 0$ (i.e., $x < 0$ or $x > 0$) since division by zero is not defined.

Step 2. Use interval notation to describe the domain of $1/x$.

The domain of $1/x$ consists of all values of x such that $x < 0$ or $x > 0$, which can be written in interval notation as

$$(-\infty, 0) \cup (0, \infty).$$

Example 31. Use interval notation to describe the domain of $\sqrt{x}$.

Step 1. Describe the domain of $\sqrt{x}$ using an inequality.

The domain of $\sqrt{x}$ includes all nonnegative real numbers (i.e., $x \geq 0$) since the square root of a negative number is not defined.

Step 2. Use interval notation to describe the domain of $\sqrt{x}$.

The domain of $\sqrt{x}$ consists of all values of x such that $x \geq 0$, which can be written in interval notation as

$$[0, \infty).$$

Example 32. Determine the domain of the function $f(x) = \sqrt{x^2 + 2x - 3}$.

Step 1. Since f is a square root function, the domain of f consists of all values of x such that

$$x^2 + 2x - 3 \geq 0$$

since the square root of a negative number is not defined.

Step 2. Solve the inequality in **Step 1**.

Recall from **Example 28** that

$$x^2 + 2x - 3 = 0$$

when $x = -3$ or $x = 1$ and

$$x^2 + 2x - 3 > 0$$

when $x < -3$ or $x > 1$. Therefore, the domain of f consists of all values of x such that $x \leq -3$ or $x \geq 1$, which can be written in interval notation as

$$(-\infty, -3] \cup [1, \infty).$$

Example 33. Determine the domain of the function $f(x) = \dfrac{x}{x^2 + 2x - 3}$.

Step 1. Since f involves the operation of division, the domain of f consists of all values of x such that

$$x^2 + 2x - 3 \neq 0$$

since division by zero is not defined.

Step 2. Find all values of x that lead to division by zero.

Solve the equation $x^2 + 2x - 3 = 0$. Recall from **Example 28** that

$$x^2 + 2x - 3 = 0$$

when $x = -3$ or $x = 1$.

Step 3. Exclude the values found in **Step 2** from the domain.

Since the only values of x that lead to division by zero are $x = -3$ and $x = 1$, the domain of f consists of all x such that $x < -3$, or $-3 < x < 1$, or $x > 1$, which can be written in interval notation as

$$(-\infty, -3) \cup (-3, 1) \cup (1, \infty).$$

Try It Yourself

Exercise 1. Rewrite $\dfrac{3}{x-1} - \dfrac{5}{2x+1}$ as a single ratio.

Exercise 2. Expand $(8x+11)(2x-5)$ using the FOIL method.

Exercise 3. Expand $5x^6(3x-4)^2$.

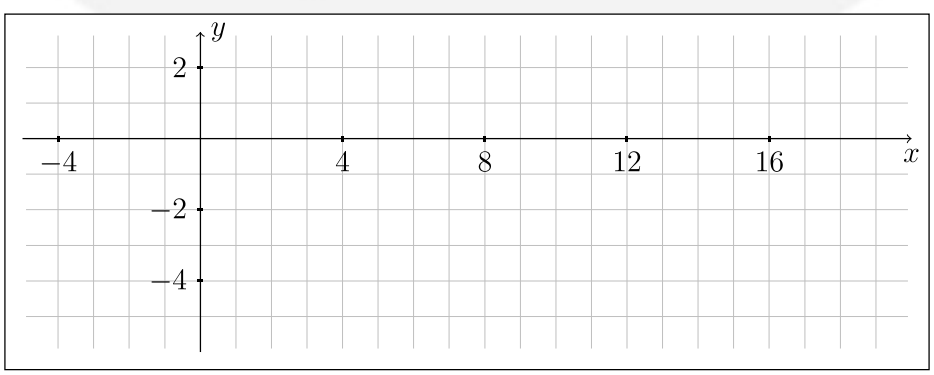
Exercise 4. Expand $(1+3x)(1-3x)(2+x)$.

Exercise 5. Factor $4x^5 - 25x^3$.

Exercise 6. Simplify $\dfrac{2x^4 - 2x^3 - 12x^2}{9x^2 - x^4}$ by factoring both the numerator and the denominator. Assume that $x \neq 0$ and $x \neq 3$.

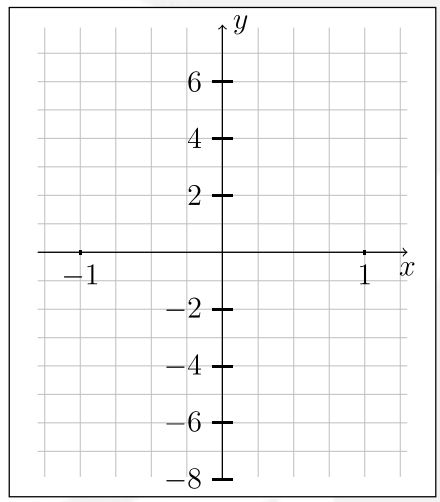
Exercise 7. Factor and simplify $10x(x+2)^5 - 5x^2(x+2)^3$.

Exercise 8. Sketch the graph of the line defined by $y + 2 = \dfrac{1}{4}(x - 5)$. Use the fact that the line is described in point-slope form.



Exercise 9. Find all values of x such that $x^2 - 16 > 0$. Write your answer using interval notation.

Exercise 10. Sketch the graph of $f(x) = 12x^2 - x - 6$ by finding the x and y-intercepts (see Example 25).



Exercise 11. Determine the domain of of each of the following functions. Write your answer using interval notation.

a) $g(x) = \dfrac{x}{12x^2 - x - 6}$

b) $h(x) = \dfrac{x}{\sqrt{12x^2 - x - 6}}$

Answers: 1) $\frac{x+8}{(x-1)(2x+1)}$ 2) $16x^2 - 18x - 55$ 3) $45x^8 - 120x^7 + 80x^6$
4) $2 + x - 18x^2 - 9x^3$ 5) $x^3(2x-5)(2x+5)$ 6) $-2(x+2)/(x+3)$
7) $5x(x+2)^3(2x^2+7x+8)$ 8) Line through the point $(5, -2)$ with a slope of $1/4$.
9) $(-\infty, -4) \cup (4, \infty)$ 10) Parabola that opens upward and goes through the points $(0, -6)$, $(3/4, 0)$, and $(-2/3, 0)$. 11) a) $(-\infty, -2/3) \cup (-2/3, 3/4) \cup (3/4, \infty)$
b) $(-\infty, -2/3) \cup (3/4, \infty)$